

LOCAL STRUCTURE OF ABELIAN COVERS

DONATELLA IACONO

ABSTRACT. We study normal finite abelian covers of smooth varieties. In particular we establish combinatorial conditions so that a normal finite abelian cover of a smooth variety is Gorenstein or locally complete intersection.

1. INTRODUCTION

Let G be a finite abelian group and let X, Y be algebraic varieties. An abelian cover X of Y , with group G is a (finite) morphism $\pi : X \longrightarrow Y$, together with a faithful action of G on X , commuting with π , such that $Y = X/G$. We will focus our attention on the case in which X is normal and Y is smooth; in this case π is flat [4, Sec. 3].

For convenience, throughout the paper we shall work over the ground field \mathbb{C} of the complex numbers. The results here apply over any algebraic closed field k and if $ch(k) \neq 0$ we only need that $\gcd(|G|, ch(k)) = 1$.

The theory of cyclic covers of algebraic surfaces was studied first by A. Comessatti in [7]. Then F. Catanese [6] studied smooth abelian covers in the case $(\mathbb{Z}_2)^2$ and R. Pardini [11] analyzed the general case. In [9] and [10] M. Manetti investigated the property to be Gorenstein for $(\mathbb{Z}_2)^n$ -covers. Recently, in [16] R. Vakil used the $(\mathbb{Z}_p)^n$ -covers to show the existence of badly-behaved deformation spaces.

The first purpose of this paper is to establish combinatorial conditions so that a normal G -cover X of a smooth variety Y ($\pi : X \longrightarrow Y$) is Gorenstein.

To give a more precise statement let us introduce the combinatorial data of the cover π at a point y in Y . Let R be the ramification locus, i.e. the set of the points of X that have non trivial stabilizer, and define the branch locus D as the image of R under π . By the theorem of purity of branch locus [19, Prop. 2], R and D are divisors. We consider the divisors with the reduced structure.

For each irreducible component T of R , let H be the subgroup of G that stabilizes T ; i.e. $H = \{h \in G : hx = x \ \forall x \in T\}$. H is called the inertia group of T and, by definition, it is a finite abelian group.

Let x be a smooth point in X and in T . By Cartan's lemma [5, Lemma 2], the inertia group H of T acts faithfully on $T_{x,X}$, leaving fixed $T_{x,T}$. Then there exists a character $\psi : H \longrightarrow \mathbb{C}^*$ such that the action on $T_{x,X}/T_{x,T}$ of an element h of H is given by the multiplication by $\psi(h)$. Since the action is faithful, ψ is a character generating the dual of H and H is a finite subgroup of \mathbb{C}^* and so it is cyclic. If $k \neq \mathbb{C}$ we cannot use Cartan's lemma, but also in this case H is cyclic and there is an associated character ψ generating the dual of H [11, Lemma 1.1 and 1.2].

Now, let E be an irreducible component of D . Since G is abelian, all the components of $\pi^{-1}(E)$ have the same inertia group and same associated character ψ . Therefore to every component of D we can associate a cyclic subgroup H of G and a character ψ generating the dual of H . For each pair (H, ψ) , with H a cyclic subgroup of G and ψ a character generating the dual of H , let $D_{H,\psi}$ be the union of all the components of D that have associated H and ψ .

Definition 1.1. Let X be a normal G -cover of a smooth variety Y . Let y be a point of Y lying on the components $D_{H_1, \psi_1}, \dots, D_{H_s, \psi_s}$ of the branch locus D . The set $\{(H_i, \psi_i)\}_{i=1, \dots, s}$ is the *combinatorial data at the point y in Y* of the cover.

Our first result is the following theorem (theorem 4.3) that relates the property to be Gorenstein to the combinatorial data.

Theorem. *Let X be a normal G -cover of a smooth variety Y and $\{(H_i, \psi_i)\}_{i=1, \dots, s}$ its combinatorial data at the point y in Y . Then the points of X over y are Gorenstein points of X if and only if there exists a character χ of G such that*

$$\chi|_{H_i} = \psi_i \quad \forall 1 \leq i \leq s.$$

It is easy to see that the existence of a character χ of G is a necessary condition. Suppose that G leaves fixed the point x in X . Since X is Gorenstein the dualizing sheaf is locally free of rank 1 [8, Ch. 21]. Then there exists a character χ of G such that the action of G on the dualizing sheaf is given by the multiplication by χ . Let y be a smooth point of X near x . Suppose that y is a smooth point of the irreducible component T of R fixed by the subgroup H of G . As already said, we can define a character ψ of H , such that the faithful action of H on $T_{y,X}/T_{y,T}$ is given by the multiplication by ψ . Therefore it is necessary that $\chi|_H = \psi$.

The previous theorem (theorem 4.3) generalizes this condition and it proves that it is necessary and sufficient condition.

Then we study locally complete intersection covers. Related to locally complete intersections there are the locally simple covers.

Definition 1.2. Let Y be a smooth variety and $\pi : X \rightarrow Y$ be a normal G -cover. Suppose that $y \in Y$ lies on the components $D_{H_1, \psi_1}, \dots, D_{H_s, \psi_s}$ of D . The G -cover is called *locally simple*, or π is *locally simple*, if the map $\oplus_i H_i \rightarrow G$ is injective, for each $y \in Y$.

Every locally simple cover is a locally complete intersection (remark 3.6).

The second purpose of this paper is to investigate when the converse holds. First of all we note that the equivalence between the properties of being locally simple and being a locally complete intersection doesn't hold for any abelian group G . Actually, example 2.6 shows the existence of a G -cover that is locally complete intersection but not locally simple.

Hence we turn our attention on $(\mathbb{Z}_p)^n$ -covers. For these covers some results already exist. In [9] Manetti proved that the equivalence holds for $(\mathbb{Z}_2)^2$ -covers [9, Prop. 1]. The following theorem (theorem 5.1) proves that the equivalence holds for each $(\mathbb{Z}_p)^n$ -covers, with p prime.

Theorem. *Let $\pi : X \rightarrow Y$ be a normal flat $(\mathbb{Z}_p)^n$ -cover with Y smooth. Then the following conditions are equivalent:*

- i): π is locally simple;
- ii): X is locally complete intersection.

This theorem is the correct version of proposition 3.25 of [10], that states the equivalence of the previous two properties and the property of being Gorenstein for $(\mathbb{Z}_2)^n$ -covers. In fact, example 2.3 shows that the property of being Gorenstein is not equivalent to the property of being locally simple for $(\mathbb{Z}_2)^n$ -covers, with $n \geq 3$. However, in [10] the result of proposition 3.25 is applied only in corollary 3.26 where Manetti used only the true equivalence between the property of being locally simple and being a locally complete intersection. Therefore this minor mistake doesn't affect the rest of the paper.

He also showed that for local simple covers the theory of deformation is easy to understand. So locally simple covers are interesting from the point of view of deformation theory.

Notations. G defines a finite abelian group and we use the additive notation. In particular, we use the notation $G = \langle a \in G : na = 0 \rangle$, with $n \in \mathbb{N}$, for a cyclic group G , generated by a , of cardinality n . So $G \cong \mathbb{Z}_n$. We also write $\langle a \rangle$ when the order is clear by the context. In general $|G|$ is the order of a group G .

$G^* = \text{Hom}(G, \mathbb{C}^*)$ is the group of characters of G (the dual of G) and we use for it the multiplicative notation.

$\xi \in \mathbb{C}^*$ stands for a primitive root of unity, whose (multiplicative) order will be specified every time.

If $a, b, c \in \mathbb{Z}$, then $a \equiv b \pmod{n}$ means that a is congruent to b modulo n , $\gcd(a, b)$ is the great common divisor between a and b and $[a]$ is the integer part.

We collect here some results that we will use in the sequel.

Theorem 1.3. (*Watanabe*) *If $f : X \longrightarrow Y$ is a flat and surjective morphism of preschemes, then X is Gorenstein if and only if Y and each fiber $f^{-1}(y)$ are Gorenstein.*

Proof. See [18, Th. 1]. □

Theorem 1.4. (*Avramov*) *Let $f : X \longrightarrow Y$ be a flat and surjective morphism of locally Noetherian schemes. Then X is a locally complete intersection if and only if the scheme $f^{-1}(y)$ is a locally complete intersection for every point $y \in Y$ and Y is a locally complete intersection.*

Proof. See [3, Cor. 2]. □

Lemma 1.5. *If (A, m) is an henselian local ring with residue field $A/m = k$ algebraically closed, then the multiplicative group $A^* = A - m$ is divisible.*

Proof. By definition A^* is divisible if and only if for each $a \in A^*$ and for each $n \in \mathbb{N}$ there exists $x \in A^*$ such that $a = x^n$. This condition is equivalent to the existence of a root in A^* of the monic polynomial $P(x) = x^n - a \in A[x]$. Let $\overline{P}(x)$ the image of $P(x)$ in $k[x]$. $\overline{P}(x)$ has a simple root \overline{a} in k and so, since A is henselian, $P(x)$ has a root a that lift \overline{a} [12, Ch. 7, Prop. 3]. □

2. THE KEY EXAMPLE

This section is devoted to the study of a particular case of G -cover $\rho : X \longrightarrow X/G$, with $X = \mathbb{C}^s/K$ and K a finite abelian group.

Let $\{(H_i, \psi_i)\}_{i=1, \dots, s}$ be a set of data, i.e. H_i are cyclic groups and ψ_i are characters generating H_i^* . If $H_i = H_j$ then we assume that $\psi_i \neq \psi_j$ in $H_i^* = H_j^*$.

Let d_i be the order of H_i , $H = \bigoplus_{i=1}^s H_i$ and K be a subgroup of H . Then we can construct an exact sequence of abelian groups

$$(1) \quad 0 \longrightarrow K \xrightarrow{\iota} H = \bigoplus_{i=1}^s H_i \xrightarrow{\nu} G \longrightarrow 0$$

where ν is the sum and G is the quotient. Moreover, suppose that each H_i injects into G .

For each element $k \in K$ we have

$$(2) \quad k \xrightarrow{\iota} \iota(k) = (h_1, \dots, h_s) \xrightarrow{\nu} \sum_{i=1}^s h_i = 0.$$

Therefore for each $k \in K - \{0\}$, $\iota(k)$ contains at least two non-zero elements $h_i \in H_i$ and $h_j \in H_j$, for $1 \leq i, j \leq s$.

First of all, we note that $H^* = \prod_{i=1}^s H_i^*$ and so, using the left-exactness of the $\text{Hom}(-, \mathbb{C}^*)$ functor [2, Prop. 2.9], we have the following exact sequence

$$(3) \quad 0 \longrightarrow G^* \xrightarrow{\nu^*} H^* \xrightarrow{\iota^*} K^*.$$

Now, we consider the action of H on \mathbb{C}^s , with coordinates (z_1, z_2, \dots, z_s) given by the characters ψ_i of the data: if $H \ni h = (h_1, \dots, h_s)$ with $h_i \in H_i$, then we define

$$h(z_1, \dots, z_s) = (\psi_1(h_1)z_1, \dots, \psi_s(h_s)z_s).$$

Each non trivial element $(0, \dots, 0, h_i, 0, \dots, 0) \in H$ acts on $\mathbb{C}^s - \{z_i = 0\}$ without fixed points. Therefore the points with all coordinates non zero have trivial stabilizer; the hyperplanes $z_i = 0$ are stabilized by $H_i \subset H$ and they correspond to the irreducible components R_i of the ramification locus R .

Let $x \in \mathbb{C}^s$ be a point of R_i . Then H_i acts on the tangent space T_{x, \mathbb{C}^s} to \mathbb{C}^s in x , leaving fixed T_{x, R_i} . Since the action of H_i is given by the multiplication by the character ψ_i , the character associated to the action of H_i on $T_{x, \mathbb{C}^s}/T_{x, R_i}$ is ψ_i itself.

The sub-ring of the invariant polynomials in $\mathbb{C}[z_1, \dots, z_s]$ is the ring $\mathbb{C}[z_1, \dots, z_s]^H = \mathbb{C}[z_1^{d_1}, \dots, z_s^{d_s}]$ (d_i is the order of H_i), since $h(z_i^{d_i}) = (\psi_i(h_i))^{d_i} z_i^{d_i} = z_i^{d_i}$.

Set $u_i := z_i^{d_i}$; then $\mathbb{C}[z_1, \dots, z_s]^H = \mathbb{C}[u_1, \dots, u_s]$ and so $\mathbb{C}^s/H \cong \mathbb{C}^s$.

Moreover the irreducible components D_i of the branch locus D are defined by the equations $u_i = 0$ and they correspond to the components D_{H_i, ψ_i} of D .

Now, we return to the exact sequence (1). Using it, we can split the action of H through an action of K followed by an action of G and we obtain the following commutative diagram

$$\begin{array}{ccc} \mathbb{C}^s & \xrightarrow{H} & \frac{\mathbb{C}^s}{H} = \mathbb{C}^s = \frac{X}{G} \\ & \searrow K & \nearrow \rho \\ & \frac{\mathbb{C}^s}{K} = X & \end{array}$$

$G\text{-cover}$

Since $X = \mathbb{C}^s/K$ is normal [5, Sec. 2] and $X/G = (\mathbb{C}^s/K)/G = \mathbb{C}^s/H$ is smooth, the map $\rho : X \rightarrow X/G$ is flat [4, Sec. 3]; in this way we have obtained a G -cover ρ .

By definition, the G -cover ρ is locally simple if K is the trivial subgroup of H .

Using (2), the action of an element $k \in K$ is determined by the image $\iota(k) = (h_1, \dots, h_s)$ and so each $k \in K - \{0\}$ acts non trivially on at least two coordinates (it isn't a pseudo-reflection).

As for H , the coordinate hyperplanes are the irreducible components of R fixed by $H_i \subset G$ (this inclusion is guaranteed by our initial assumption H_i injects in G). Analogously the irreducible components of the branch locus D of ρ are the D_i defined by the equations $u_i = 0$.

The coordinate ring of $X = \mathbb{C}^s/K$ is $\mathbb{C}[z_1, \dots, z_s]^K \subset \mathbb{C}[z_1, \dots, z_s]$. Since K is a finite abelian group that acts diagonally, $\mathbb{C}[z_1, \dots, z_s]^K$ is generated by invariant monomials.

Let $N_K^s = \{(\alpha_1, \dots, \alpha_s) \in \mathbb{N}^s \mid z_1^{\alpha_1} \cdots z_s^{\alpha_s} \text{ is an invariant monomial for } K\}$. Then we can write $\mathbb{C}[z_1, \dots, z_s]^K = \bigoplus_{(\alpha_1, \dots, \alpha_s) \in N_K^s} z_1^{\alpha_1} \cdots z_s^{\alpha_s} \mathbb{C}$.

A monomial $z_1^{\alpha_1} \cdots z_s^{\alpha_s}$ lies in N_K^s if and only if $\iota^*(\Pi_i \psi_i^{\alpha_i}) = 1 \in K^*$. Actually, each element of $K \leq H = \bigoplus_{i=1}^s H_i$ is of the form $k = (k_1, \dots, k_s)$ and so $k(z_1^{\alpha_1} \cdots z_s^{\alpha_s}) = z_1^{\alpha_1} \cdots z_s^{\alpha_s} (\forall k)$ if and only if $\psi_1(k_1)^{\alpha_1} \cdots \psi_s(k_s)^{\alpha_s} = z_1^{\alpha_1} \cdots z_s^{\alpha_s}$ that is $\Pi_i \psi_i^{\alpha_i} = 1 \in K^*$.

By the exact sequence (3), this is equivalent to the existence of a character $\chi \in G^*$ such that $\nu^*(\chi) = \Pi_i \psi_i^{\alpha_i}$. We notice that the natural numbers α_i are such that $\chi|_{H_i} = \psi_i^{\alpha_i}$, for each i .

Therefore $\mathbb{C}[z_1, \dots, z_s]^K = \bigoplus_{(\alpha_1, \dots, \alpha_s) \in N_K^s} z_1^{\alpha_1} \cdots z_s^{\alpha_s} \mathbb{C} = \bigoplus_{\chi \in G^*} \mathcal{A}_\chi$ where the \mathcal{A}_χ are

defined as follows.

Since the groups H_i^* have order d_i , $\mathcal{A}_1 = \mathbb{C}[z_1^{d_1}, \dots, z_s^{d_s}] = \mathbb{C}[u_1, \dots, u_s]$; moreover \mathcal{A}_χ are free \mathcal{A}_1 -module of rank 1. In fact, if the character $\chi = \prod_i \psi_i^{\alpha_i} \in G^*$ is associated to the invariant monomial $z_1^{\alpha_1} \cdots z_s^{\alpha_s}$, then the other invariant monomials $z_1^{\beta_1} \cdots z_s^{\beta_s}$ with the same associated character $\chi = \prod_i \psi_i^{\alpha_i}$ are such that $\beta_i \equiv \alpha_i \pmod{d_i}$, for each $0 \leq i \leq s$. Hence, if $z_1^{\alpha_1} \cdots z_s^{\alpha_s}$ is the unique invariant monomials associated to χ , with $0 \leq \alpha_i \leq d_i - 1$, we write $w_\chi = z_1^{\alpha_1} \cdots z_s^{\alpha_s}$ and so w_χ is a generator of \mathcal{A}_χ as free \mathcal{A}_1 -module, i.e. $\mathcal{A}_\chi = \langle w_\chi = z_1^{\alpha_1} \cdots z_s^{\alpha_s} \rangle$.

Now we analyze the multiplicative structure of $\mathbb{C}[z_1, \dots, z_s]^K$.

Given $w_\chi = z_1^{\alpha_1} \cdots z_s^{\alpha_s}$ and $w_{\chi'} = z_1^{\alpha'_1} \cdots z_s^{\alpha'_s}$, we define

$$(4) \quad \varepsilon_{\chi, \chi'}^i := \left\lfloor \frac{\alpha_i + \alpha'_i}{d_i} \right\rfloor.$$

As already observed, all the α_i and α'_i satisfy $\chi|_{H_i} = \psi_i^{\alpha_i}$ and $\chi'|_{H_i} = \psi_i^{\alpha'_i}$.

Fix the attention on the fiber of ρ over $0 \in \mathbb{C}^s/H$: 0 lies on the components D_{H_i, ψ_i} for each $i = 1, \dots, s$ and so $u_i = 0$. Then $w_\chi \cdot w_{\chi'} = 0$ if there exists i such that $\alpha_i + \alpha'_i \geq d_i$, otherwise $w_\chi \cdot w_{\chi'} = w_{\chi \cdot \chi'}$.

Hence the product in $\mathbb{C}[z_1, \dots, z_s]^K$ is given by

$$(5) \quad w_\chi \cdot w_{\chi'} = z_1^{\alpha_1 + \alpha'_1} \cdots z_s^{\alpha_s + \alpha'_s} = w_{\chi \cdot \chi'} \prod_{i=1}^s u_i^{\varepsilon_{\chi, \chi'}^i}.$$

2.1. When is the key example Gorenstein? Now we investigate when X of the G -cover $\rho : \mathbb{C}^s/K = X \longrightarrow X/G = \mathbb{C}^s/H = \mathbb{C}^s$ is Gorenstein, establishing combinatorial conditions on the data $\{(H_i, \psi_i)\}_{i=1, \dots, s}$.

Since $X = \mathbb{C}^s/K$, we can apply an useful result due to Watanabe [17, Sec. 2, Th. 1].

Theorem 2.1. (Watanabe) *Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field k and G be a finite subgroup of $GL(n, k)$ with $\gcd(|G|, \text{ch}(k)) = 1$ if $\text{ch}(k) \neq 0$. We also assume that G contains no pseudo reflections. Then R^G is Gorenstein if and only if $G \leq SL(n, k)$.*

In our example the action of K just depends on the action of H and so it is strictly related on the data $\{(H_i, \psi_i)\}_{i=1, \dots, s}$. Therefore we can rewrite the previous theorem 2.1 in term of combinatorial conditions on the characters ψ_i .

More precisely, we have associated to each element $k \in K$ a diagonal matrix of the action on \mathbb{C}^s , whose entries are the characters ψ_i valuated on k . Therefore we need to know when a product of characters ψ_i is 1 for each element $k \in K$. Using the exact sequence (3), the previous condition is related to the existence of a character of G that

lifts the characters ψ_i . In conclusion we have proved the following proposition that is a particular case of the first theorem.

Proposition 2.2. *Let $\{(H_i, \psi_i)\}_{i=1, \dots, s}$ be a set of data and K a subgroup of $H = \bigoplus_{i=1}^s H_i$.*

Let G be the quotient and suppose that each H_i injects in G . \mathbb{C}^s/K is Gorenstein if and only if there exists a character χ of G such that

$$(6) \quad \chi|_{H_i} = \psi_i \quad \forall 1 \leq i \leq s.$$

2.2. Examples. Now, we explicitly analyze some G -covers. Fixing a finite abelian group G , we can consider a set of combinatorial data $\{(H_i, \psi_i)\}_{i=1, \dots, s}$, with H_i cyclic subgroups of G (so each one injects in G) and ψ_i a character generating H_i^* . If $H_i = H_j$ then we assume that $\psi_i \neq \psi_j$ in $H_i^* = H_j^*$. Suppose that $\nu : H = \bigoplus_{i=1}^s H_i \rightarrow G$ is surjective. Also in this case we can construct an exact sequence as (1) with K the kernel of ν .

Example 2.3. Let $G \cong (\mathbb{Z}_2)^3$ with standard generators e_i and let $H_i = \langle e_i \rangle$, for $i = 1, 2, 3$, and $H_4 = \langle e_1 + e_2 + e_3 \rangle$ with associated character ψ_1, ψ_2, ψ_3 and φ , respectively ($\psi_i(e_i) = \xi, \varphi(e_1 + e_2 + e_3) = \xi$, with $\xi \in \mathbb{C}^*$ and $\xi^2 = 1$).

In this example, $H = (\mathbb{Z}_2)^4$ and for $k \in K - \{0\}$ we have $\iota(k) = (e_1, e_2, e_3, e_1 + e_2 + e_3) \in H$. The diagram associated to the action of H on \mathbb{C}^4 is the following

$$\begin{array}{ccc} \mathbb{C}^4 & \xrightarrow{H} & \frac{\mathbb{C}^4}{H} = \frac{X}{(\mathbb{Z}_2)^3} \\ & \searrow K & \nearrow (\mathbb{Z}_2)^3\text{-cover} \\ & \frac{\mathbb{C}^4}{\pm Id} = X & \end{array}$$

with actions given by:

- $h \in H$ is of the form $h = t_1 e_1 + t_2 e_2 + t_3 e_3 + t_4 (e_1 + e_2 + e_3)$, with $t_i = 0, 1$; so $h(z_1, z_2, z_3, z_4) = ((-1)^{t_1} z_1, (-1)^{t_2} z_2, (-1)^{t_3} z_3, (-1)^{t_4} z_4)$.
- $k \in K - \{0\}$ acts as: $k(z_1, z_2, z_3, z_4) = (-z_1, -z_2, -z_3, -z_4)$.
- $g \in G$ is of the form $g = s_1 e_1 + s_2 e_2 + s_3 e_3$, with $s_i = 0, 1$; so $g(w_1, w_2, w_3, w_4) = ((-1)^{s_1} w_1, (-1)^{s_2} w_2, (-1)^{s_3} w_3, w_4)$.

The $(\mathbb{Z}_2)^3$ -cover $\rho : X \rightarrow X/(\mathbb{Z}_2)^3$ is not locally simple ($K \neq \{0\}$) but, since $K \leq SL(4, \mathbb{C})$, X is Gorenstein by theorem 2.1.

Remark 2.4. In this example X cannot be a local complete intersection.

Let us recall a theorem due to Schlessinger [15, Sec. 10] or [13, Sec. 3].

Theorem 2.5. (Schlessinger) *Quotient singularities which are non singular in codimension two are rigid, that is every infinitesimal deformation is trivial.*

We can apply this theorem to the cover $\mathbb{C}^4 \rightarrow \mathbb{C}^4/K$. The quotient \mathbb{C}^4/K has a singularity only at the origin (0 is the only fixed point) and its codimension is 4. Therefore every infinitesimal deformation of $X = \mathbb{C}^4/K$ is trivial. On the other hand, singular complete intersections are not infinitesimally rigid (cf. [1]). Therefore X cannot be a local complete intersection.

Example 2.6. Let $G \cong \mathbb{Z}_{pqr}$, with $p < q < r$ prime numbers. Let $\xi \in \mathbb{C}^*$ be a primitive root of 1 of order pqr , and G generated by a . Let $H_1 \cong \mathbb{Z}_{pr} = \langle qa \rangle$ and $H_2 \cong \mathbb{Z}_{pq} = \langle ra \rangle$, $\psi_1(qa) = \xi^{q\alpha}$, with $\gcd(\alpha, pr) = 1$, and $\psi_2(ra) = \xi^{r\beta}$, with $\gcd(\beta, pq) = 1$. In this example, $H = H_1 \oplus H_2$ and $K \cong \mathbb{Z}_p$; so we have the following commutative diagram

$$\begin{array}{ccc}
\mathbb{C}^2 & \xrightarrow{H} & \frac{\mathbb{C}^2}{H} \\
& \searrow K & \nearrow \mathbb{Z}_{pqr}\text{-cover} \\
& & \frac{\mathbb{C}^2}{K}
\end{array}$$

Applying proposition 2.2, \mathbb{C}^s/K is Gorenstein if and only if $\psi_1|_K = \psi_2|_K$ or equivalently $\alpha \equiv \beta \pmod{p}$.

Moreover, if ψ_1 and ψ_2 satisfy this condition then \mathbb{C}^2/K is a local complete intersection \mathbb{Z}_{pqr} -cover that isn't locally simple ($K \cong \mathbb{Z}_p$). In fact, if $\alpha \equiv \beta \pmod{p}$ the action of an element $k \in K$ on \mathbb{C}^2 , with coordinates (z_1, z_2) , is of the form $k(z_1, z_2) = (z_1\eta, z_2\eta^{-1})$, where $\eta \in \mathbb{C}^*$ is a p -root of 1. Therefore \mathbb{C}^2/K has a rational double point at 0 (of type A_{p-1}) and it is a local complete intersection.

Example 2.7. Let $G \cong \mathbb{Z}_{p^n}$, with p prime. All the subgroups of G are cyclic p -groups. Let $\{(H_i, \psi_i)\}_{i=1, \dots, s}$ be a set of combinatorial data. Then $H = \oplus_i H_i$ is the sum of p -groups with at least one isomorphic to G (to have a surjection). Relabelling if necessary the combinatorial data, we can suppose that $H = H_1 \oplus \dots \oplus H_s$, with $H_s \cong G$. Therefore ψ_s is a character of G and so, according to proposition 2.2, the condition for $X = \mathbb{C}^s/K$ to be Gorenstein becomes $\psi_s|_{H_i} = \psi_i$, for each $1 \leq i \leq s-1$. Moreover, the group G is cyclic and so there exists only one cyclic subgroup of \mathbb{Z}_{p^n} of order p^i , for each $i = 1, \dots, n$. For this reason the cyclic groups H_i must have different orders and so $s \leq n$. Then we can reorganize them so that $H_1 < H_2 < \dots < H_s = G$ and the following restrictions must match, i.e: $\psi_s|_{H_j} = \psi_j = \psi_{j+1}|_{H_j}$, for each $1 \leq j \leq s-1$.

Example 2.8. Let $G \cong (\mathbb{Z}_p)^n$ (p prime) and $\{(H_i, \psi_i)\}_{i=1, \dots, s}$ be a set of combinatorial data. In this case the only admissible cyclic subgroups of G have order p ($H_i \cong \mathbb{Z}_p$); therefore the exact sequence (1) can be written as

$$(7) \quad 0 \longrightarrow K \longrightarrow H = \bigoplus_{i=1}^s \mathbb{Z}_p \longrightarrow G \longrightarrow 0$$

with $s \geq n$.

Remark 2.9. If $s = n$ then $H = G$ and we have a locally simple cover.

If $s > n$ and the $(\mathbb{Z}_p)^n$ -cover $\rho : \mathbb{C}^s/K = X \longrightarrow X/(\mathbb{Z}_p)^n$ is Gorenstein, then $H_i \neq H_j$ for each $1 \leq i, j \leq s$ and $i \neq j$. In fact, assume that $H_i = H_j$; then by definition $\psi_i \neq \psi_j$. If X is Gorenstein then there exists a character χ of G such that $\psi_i = \chi|_{H_i} = \chi|_{H_j} = \psi_j$ that is a contradiction. In particular $H_i \cap H_j = \{0\}$.

Therefore, looking at the sequence (2), for each element $k \in K - \{0\}$ we have $\iota(k) = (h_1, \dots, h_s)$ with at least three elements h_j different from zero, with $1 \leq j \leq s$. This implies that each element $k \in K - \{0\}$ acts non trivially at least on three coordinates of the points in \mathbb{C}^s .

This fact will be used in the proof of the main theorem (theorem 5.1).

3. STRUCTURE OF ABELIAN COVER

In [11], Pardini completely described the structure of normal abelian covers of smooth complete algebraic varieties (theorem 3.1); she also gave conditions so that a normal G -cover of a smooth variety is smooth (theorem 3.5). In this section we introduce some notations to recall these theorems. A detailed description can be found in [11].

Let $\pi : X \longrightarrow Y$ be an abelian G -cover with X normal and Y smooth; then π is flat and $\pi_* \mathcal{O}_X$ is locally free [4]. The action of G on X induces the splitting $\pi_* \mathcal{O}_X =$

$\bigoplus_{\chi \in G^*} L_\chi^{-1}$, with L_χ^{-1} line bundle, on which G acts via the character χ , and L_1 isomorphic to \mathcal{O}_Y .

Let $D_{H,\psi}$ be the union of all the components of D that have associated the same subgroup H and character ψ . L_χ and $D_{H,\psi}$ are called the *building data of the cover*.

Suppose $\chi, \chi' \in G^*$. Then $\chi|_H$ and $\chi'|_H$ belong to H^* and so there exist i_χ and $i_{\chi'}$ in $\{0, 1, \dots, |H| - 1\}$, such that $\chi|_H = \psi^{i_\chi}$ and $\chi'|_H = \psi^{i_{\chi'}}$. Finally, let $\varepsilon_{\chi, \chi'}^{H, \psi} = \lfloor \frac{i_\chi + i_{\chi'}}{|H|} \rfloor$. Using the above notations, we can state the following theorem due to Pardini.

Theorem 3.1. (*Pardini*) *Let G be an abelian group. Let Y be a smooth variety, X a normal one and let $\pi : X \rightarrow Y$ be an abelain cover with group G . Then the following set of linear equivalences is satisfied by the building data of the cover:*

$$(8) \quad L_\chi + L_{\chi'} = \sum_{H, \psi} \varepsilon_{\chi, \chi'}^{H, \psi} D_{H, \psi}$$

Conversely, to any set of data L_χ , $D_{H, \psi}$ satisfying (8) we can associate an abelian cover $\pi : X \rightarrow Y$ in a natural way. Whenever the cover so constructed is normal, L_χ and $D_{H, \psi}$ are its building data.

Moreover, if Y is complete, then the building data determine the cover $\pi : X \rightarrow Y$ up to isomorphism of Galois cover.

Proof. See [11, th. 2.1]. □

Remark 3.2. About the existence of the cover, let L_χ and $D_{H, \psi}$ ($L_1 = \mathcal{O}_Y$) satisfy relations (8). Let $\sigma_{H, \psi} \in \mathcal{O}(D_{H, \psi})$ be a section defining $D_{H, \psi}$ and $A = \bigoplus_{\chi \in G^*} L_\chi^{-1}$. Using

(8), the formula

$$(9) \quad \mu_{\chi, \eta} = \prod_{H, \psi} \sigma_{H, \psi}^{\varepsilon_{\chi, \eta}^{H, \psi}}$$

defines an associative multiplication on A . G acts on each L_χ^{-1} by multiplication by χ and so we can extend this action to an action of G over A , compatible with the multiplicative structure. Therefore $X = \text{Spec } A$ is a G -cover of Y .

The equation (5) of section 2 is the explicit version of (9) for the key example.

Proposition 3.3. *Let $\pi : X \rightarrow Y$ and $\pi' : X \rightarrow Y$ be G -covers with the same building data. For each point $y \in Y$ there exists an étale neighborhood U_y over which the covers π and π' are isomorphic.*

Proof. Assume that $\pi : X \rightarrow Y$ and $\pi' : X \rightarrow Y$ are two G -covers with the same building data. Let $A = \bigoplus_{\chi \in G^*} L_\chi^{-1}$ and denote with $\mu_{\chi, \eta}$ and $\mu'_{\chi, \eta}$ the two algebra struc-

tures on A corresponding to π and π' . Since the relations (8) must be satisfied, by (9) we can conclude that $\mu_{\chi, \eta}$ and $\mu'_{\chi, \eta}$ have the same divisors, for each $\chi, \eta \in G^*$; therefore, for each $\chi, \eta \in G^*$, there exists a section $c_{\chi, \eta} \in \Gamma(Y, \mathcal{O}_Y^*)$ such that

$$(10) \quad \mu_{\chi, \eta} = c_{\chi, \eta} \mu'_{\chi, \eta}.$$

Since $\mu_{\chi, \eta}$ and $\mu'_{\chi, \eta}$ are associative multiplications, the sections $c_{\chi, \eta}$ satisfy the following identity

$$(11) \quad c_{\eta, \tau} c_{\eta\tau, \chi} = c_{\chi, \eta} c_{\chi\eta, \tau} \quad \forall \chi, \eta, \tau \in G^*.$$

Let φ_χ be automorphisms of the invertible sheaves L_χ . If w_χ generates L_χ then there exists $a_\chi \in \Gamma(Y, \mathcal{O}_Y^*)$ such that $\varphi_\chi(w_\chi) = a_\chi w_\chi$.

By definition a G -isomorphism $\varphi : (A, \mu) \rightarrow (A, \mu')$ of the G -algebras A with multiplication μ and A with μ' , is such that $\varphi(w_\chi \cdot w_\eta) = \varphi(w_\chi) \cdot \varphi(w_\eta)$. This condition is

satisfied if and only if $a_{\chi\eta}\mu_{\chi,\eta}w_{\chi\eta} = a_{\chi}a_{\eta}\mu'_{\chi,\eta}w_{\chi\eta}$. Therefore, using the identity (10), to produce a G -isomorphism φ we have to show the existence of elements $a_{\chi} \in \Gamma(Y, \mathcal{O}_Y^*)$, for each $\chi \in G^*$, such that

$$(12) \quad c_{\chi,\eta} = \frac{a_{\chi}a_{\eta}}{a_{\chi\eta}} \quad \forall \chi, \eta \in G^*.$$

Fix the attention over a point $y \in Y$. To prove the existence of the elements a_{χ} we use the cohomology of the group G^* with coefficients in $B = \mathcal{O}_{y,Y}^*$ (with multiplicative notation) considered as trivial G^* -module.

The elements $c_{\chi,\eta}$ can be considered as elements of $C^2(G^*, \mathcal{O}_{y,Y}^*)$ by $c(\chi, \eta) = c_{\chi,\eta} \in \mathcal{O}_{y,Y}^*$. The associative relations (11) for $c_{\chi,\eta}$ is exactly the 2-cocycle condition for $c(\chi, \eta)$ (see [14, Ch. VII, sec. 3]). Therefore $c_{\chi,\eta} \in H^2(G^*, \mathcal{O}_{y,Y}^*)$.

In general $H^2(G^*, \mathcal{O}_{y,Y}^*) \neq 0$. Let $\widetilde{\mathcal{O}_{y,Y}}$ be the henselianization of $\mathcal{O}_{y,Y}$. By lemma 1.5, $\widetilde{\mathcal{O}_{y,Y}}^*$ is divisible and so $H^q(G^*, \widetilde{\mathcal{O}_{y,Y}}^*) = 0$ for each $q \geq 1$. This implies that there exists $h \in C^1(G^*, \widetilde{\mathcal{O}_{y,Y}}^*)$ such that $c = dh$, or better $c_{\chi,\eta} = \frac{h(\chi)h(\eta)}{h(\chi\eta)}$ for each $\chi, \eta \in G^*$.

The henselianization $\widetilde{\mathcal{O}_{y,Y}}$ of $\mathcal{O}_{y,Y}$, is an inductive limit of local rings (B_i, m_i) with B_i étale over $\mathcal{O}_{y,Y}$ (see [12, Th. 1, pag. 87]). Then there exists a local ring (B, m) with B étale over $\mathcal{O}_{y,Y}$, such that $h \in C^1(G^*, B^*)$. Let $a_{\chi} = h(\chi)$ for each $\chi \in G^*$. Then the identity (12) holds and so in the étale neighborhood $U = \text{Spec } B$ of y the two cover π and π' are isomorphic. \square

Remark 3.4. By theorem 3.1 and proposition 3.3, the fiber of a G -cover $\pi : X \rightarrow Y$ over a point $y \in Y$ just depends on the combinatorial data at y of π .

Now suppose that L_{χ} are invertible sheaves and $D_{H,\psi}$ are divisors, on a variety Y , satisfying conditions (8). Let $\mathfrak{M}/\mathfrak{M}^2$ be the cotangent space. The following theorem, due to Pardini, determines the conditions on the building data so that an abelian cover of a smooth variety is smooth.

Theorem 3.5. (Pardini) *Assume that Y is a smooth variety and that $\pi : X \rightarrow Y$ is the G -cover of X with building data L_{χ} 's and $D_{H,\psi}$'s given by the previous theorem 3.1. Then X is non singular over a point $y \in Y$ if and only if one of the following conditions holds:*

- i): y is not a branch point of π ;
- ii): y belongs to one component Δ of D and y is a smooth point of Δ ;
- iii): y lies on components $D_{H_1,\psi_1}, \dots, D_{H_r,\psi_r}$ of D and:
 - a): the map $H_1 \oplus \dots \oplus H_r \rightarrow G$ is an injection,
 - b): let b_i be a local equation for D_{H_i,ψ_i} around y , $i = 1, \dots, r$. Then the subspaces of $\mathfrak{M}/\mathfrak{M}^2$ generated by db_1, \dots, db_r has dimension r .

Proof. See [11, Prop. 3.1]. \square

Remark 3.6. Pardini proved that a locally simple cover is locally complete intersection [11, Prop. 2.1]. Theorem 5.1 proves that the converse is true for $(\mathbb{Z}_p)^n$ -covers, for each prime p and natural number n and example 2.6 proves that it isn't true in general.

4. PROOF OF THE FIRST THEOREM (GENERAL CASE)

In this section we prove the theorem that relates the property to be Gorenstein to the combinatorial data of a cover. The main idea is to reduce the general case of a G -cover $\pi : X \rightarrow Y$ to a G -cover of the form $\rho : \mathbb{C}^s/K \rightarrow (\mathbb{C}^s/K)/G$ and then apply the proposition 2.2 of the previous section.

Let $\pi : X \rightarrow Y$ be a G -cover of algebraic varieties with X normal and Y smooth. Since G is a finite group, the fibers are 0-dimensional and their cardinalities divide the

order of G . The property to be Gorenstein is local and so we restrict our attention on X over a fixed point y in Y .

By theorem 1.3, X is Gorenstein over y if and only if the fiber of the π -cover over y is Gorenstein; therefore, from now on, we will restrict our attention just on the fibers.

Suppose that $y \in Y$ lies on the components $D_{H_1, \psi_1}, \dots, D_{H_s, \psi_s}$ of the branch locus D . Therefore the set $\{(H_i, \psi_i)\}_{i=1, \dots, s}$ is the combinatorial data at the point $y \in Y$ of π and, by remark 3.4, it determines the fiber of π over y .

Let H be the direct sum of the cyclic groups H_i , i.e. $H = \bigoplus_{i=1}^s H_i$. Then we have a map ν from H to G (ν is the sum).

It is not restrictive to assume that ν is a surjection, since it is possible to factorize the cover π near y as the composition of a totally ramified cover (i.e. inertia groups generate G) followed by an étale map in the following way. Let M be the subgroup of G generated by the inertia groups and T the quotient G/M . Then we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{G} & \frac{X}{G} = Y = \frac{Z}{T} \\ & \searrow M & \nearrow T \\ & \frac{X}{M} = Z. & \end{array}$$

where $X \rightarrow Z$ is a totally ramified cover and $Z \rightarrow \frac{Z}{T}$ is étale [6, pag. 487].

If K is the kernel of ν , as in section 2, we can consider the exact sequence of abelian groups

$$0 \longrightarrow K \xrightarrow{\iota} H = \bigoplus_{i=1}^s H_i \xrightarrow{\nu} G \longrightarrow 0.$$

and the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^s & \xrightarrow{H} & \frac{\mathbb{C}^s}{H} = \mathbb{C}^s \\ & \searrow K & \nearrow \rho \\ & \frac{\mathbb{C}^s}{K} & \text{G-cover} \end{array}$$

Since H_i are cyclic subgroups of G , they inject in G . In this way we have obtained another G -cover ρ .

Now we prove a fundamental lemma for the proof of the theorem.

Lemma 4.1. *The fiber of the π -cover over $y \in Y$ is isomorphic to the fiber of the ρ -cover over $0 \in \mathbb{C}^s/H$.*

Proof. Let us consider the G -cover $\pi : X \rightarrow Y$. By hypothesis, $y \in Y$ lies on the components $D_{H_1, \psi_1}, \dots, D_{H_s, \psi_s}$ of the branch locus D . Therefore $\{(H_i, \psi_i)\}_{i=1, \dots, s}$ is the set of the combinatorial data at the point $y \in Y$ of π ; moreover, by theorem 3.1, the building data of this cover satisfy the relation (8) with $\varepsilon_{\chi, \chi'}^{H, \psi} = \varepsilon_{\chi, \chi'}^{H_j, \psi_j} = \left[\frac{i_\chi + i_{\chi'}}{|H_j|} \right]$.

Now, let us examine the G -cover $\rho : \mathbb{C}^s/K \rightarrow \mathbb{C}^s/H$ constructed above. We analyzed this type of cover in section 2. In this case $0 \in \mathbb{C}^s/H$ lies on the components $D_j = D_{H_j, \psi_j}$, for $j = 1, \dots, s$ and so the combinatorial data $\{(H_j, \psi_j)\}$ of ρ at $0 \in \mathbb{C}^s/H$ is the same of the π -cover in y . Moreover, the building data of the ρ -cover satisfy equation (8) with coefficient $\varepsilon_{\chi, \chi'}^j$ and by definition $\varepsilon_{\chi, \chi'}^{H_j, \psi_j} = \varepsilon_{\chi, \chi'}^j$.

Applying remark 3.4 we can conclude that the fiber of the ρ -cover in 0 is isomorphic to the fiber of the π -cover in y . \square

Definition 4.2. Let $\pi : X \rightarrow Y$ be a G -cover with X normal and Y smooth variety. The ρ cover defined above is called the *combinatorial cover associated to π in y* .

Now we can prove the theorem.

Theorem 4.3. *Let X be a normal G -cover of a smooth variety Y and $\{(H_i, \psi_i)\}_{i=1, \dots, s}$ its combinatorial data at the point y . Then the points of X over y are Gorenstein points of X if and only if there exists a character χ of G such that*

$$(13) \quad \chi|_{H_i} = \psi_i \quad \forall 1 \leq i \leq s.$$

Proof. By theorem 1.3, X is Gorenstein over y if and only if the fiber $\pi^{-1}(y)$ is Gorenstein. Let ρ be the combinatorial cover associated to π in y . Applying the previous lemma 4.1, the fiber $\pi^{-1}(y)$ is Gorenstein if and only if the fiber $\rho^{-1}(0)$ is Gorenstein. Applying again theorem 1.3, $\rho^{-1}(0)$ is Gorenstein if and only if \mathbb{C}^s/K is Gorenstein over 0. Now we can apply proposition 2.2 and conclude the proof. \square

5. PROOF OF THE MAIN THEOREM

In this section we prove the equivalence between the properties of being locally simple and being a locally complete intersection for $(\mathbb{Z}_p)^n$ -covers, for each prime p and natural numbers n .

Theorem 5.1. *Let $\pi : X \rightarrow Y$ be a normal flat $(\mathbb{Z}_p)^n$ -cover with Y smooth. Then the following conditions are equivalent:*

- i): π is locally simple;
- ii): X is locally complete intersection.

Proof. i) \Rightarrow ii). It follows from definition 1.2 of locally simple and from remark 3.6.
ii) \Rightarrow i). Suppose that the cover is not locally simple. By definition there exists a point $y \in Y$ such that, if $\{(H_i, \psi_i)\}_{i=1, \dots, s}$ is its combinatorial data, the map $\nu : H = \bigoplus_i H_i \rightarrow G$ isn't injective. Therefore the kernel K of the associated combinatorial cover ρ is not trivial.

By hypothesis, X is a locally complete intersection and so X is Gorenstein over y . Therefore by theorem 4.3, the characters of the combinatorial data at the point y of the π -cover satisfy the identity (13). Then the combinatorial data of the associated combinatorial cover ρ at the point $0 \in \mathbb{C}^s/H$ satisfy the same identity. We studied the $(\mathbb{Z}_p)^n$ -covers in example 2.8 and remark 2.9 shows that the codimension of the singularities is at least three. Hence applying theorem 2.5 as in remark 2.4, \mathbb{C}^s/K is infinitesimally rigid and so it isn't a complete intersection. Applying theorem 1.4, the fiber $\rho^{-1}(0)$ is not a locally complete intersection and so, by lemma 4.1, the fiber of π over y is not a locally complete intersection. Therefore, applying again theorem 1.4, X is not a complete intersection in y and this is a contradiction. \square

Remark 5.2. At this stage we could try to extend the previous equivalence to each G -cover. As already observed, the implication i) \Rightarrow ii) still holds for each G -cover (remark 3.6). In section 2, example 2.6 shows the existence of a local complete intersection G -cover that isn't locally simple and so ii) \Rightarrow i) doesn't hold in general. Probably for the general case, we have to generalize the notion of locally simple cover to have an equivalence with locally complete intersection.

Acknowledgments. I wish to thank M. Manetti for having introduced me to this problem and for all remarks, suggestions and encouragements; several ideas of this paper

are grown under his influence. I am very grateful to the referees for their improvements in the presentation of the paper.

REFERENCES

- [1] M. Artin, *Deformations of Singularities*, Tata Institute of Fundamental Research, Bombay, 1976.
- [2] M. F Atiyah, I. G. Macdonald *Introduction to Commutative Algebra*, Addison-Wesley Publishing Company, 1969.
- [3] L. L. Avramov, *Flat morphisms of complete intersection*, Soviet Math. Dokl., **16** (1975), 1413-1417.
- [4] T. Berry, *Infinitesimal deformations of cyclic covers*, Acta Cientifica Venezolana, **35** (1984), 177-183.
- [5] H. Cartan, *Quotient d'un espace analytique par un groupe d'automorphismes*, Algebraic Geometry and topology: A symposium in honour of S. Lefschetz, Princeton Math. Series **12** (1957), 90-112.
- [6] F. Catanese, *On the moduli spaces of surfaces of general type*, J. Differential Geom., **19** (1984), 483-515.
- [7] A. Comessatti, *Sulle superfici multiple cicliche*, Rend. Sem. Mat. Univ. Padova, **1** (1930), 1-45.
- [8] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate texts in mathematics, **150**, Springer-Verlag, New York/Berlin, 1995.
- [9] M. Manetti, *On some components of moduli space of surfaces of general type*, Comp. Math., **92** (1994), 285-297.
- [10] M. Manetti, *On the moduli space of diffeomorphic algebraic surfaces*, Invent. math., **143** (2001), 29-76.
- [11] R. Pardini, *Abelian covers of algebraic varieties*, J. Reine Angew. Math., **417** (1991), 191-213.
- [12] M. Raynaud, *Anneaux locaux henseliens*, Lecture Notes in Mathematics, **169**, Springer-Verlag, New York/Berlin, 1970.
- [13] M. Schlessinger, *On rigid singularities*, Rice Univ. Stud., **59** vol.1 (1973), 147-162.
- [14] J.P. Serre, *Corps Locaux*, Hermann, Paris, (1968).
- [15] J. Stevens, *Deformations of Singularities*, Lecture Notes in Mathematics, **1811**, Springer-Verlag, New York/Berlin, 2003.
- [16] R. Vakil, *Murphy's law in algebraic geometry: badly-behaved deformation spaces*, arXiv:math.AG/0411469v1.
- [17] K. Watanabe, *Certain invariant subrings are Gorenstein II*, Osaka J. Math., **11** (1974), 379-388.
- [18] K. Watanabe, T. Ishikawa, S. Tachibana and K. Otsuka, *On tensor products of Gorenstein rings*, J. Math. Kyoto Univ., **9** (1969), 413-423.
- [19] O. Zariski, *On the purity of branch locus of algebraic functions*, Proc. Nat. Acad. Sci. U.S.A., **44** (1958), 791-796.

DIPARTIMENTO DI MATEMATICA "GUIDO CASTELNUOVO",
 UNIVERSITÀ DI ROMA "LA SAPIENZA",
 P.LE ALDO MORO 5, I-00185 ROMA, ITALY.
E-mail address: `iacono@mat.uniroma1.it`